Exchange economy with generic Cobb-Douglas

Given the following utility functions:

$$U^A = (x_1^A)^\alpha (x_2^A)^\beta$$
$$U^B = (x_1^B)^\gamma (x_2^B)^\delta$$

and the initial endowments:

$$m^A = p_1\omega_1^A + p_2\omega_2^A$$
$$m^B = p_1\omega_1^B + p_2\omega_2^B$$

1. Solve for $\frac{p_1}{p_2}$, given that the Cobb-Douglas marshallian demands are:

$$U = Kx^\alpha y^\beta$$
$$s.t.: m = P_x X + P_y Y$$
$$x^* = \frac{m}{P_x}\frac{\alpha}{\alpha+\beta}$$
$$y^* = \frac{m}{P_y}\frac{\beta}{\alpha+\beta}$$

2. Find the numerical value for p_1/p_2 given:

$$\alpha = \beta = \gamma = \delta = 1$$
$$\omega_1^A = 10$$
$$\omega_2^A = 15$$
$$\omega_1^B = 7$$
$$\omega_2^B = 20$$

Solution

1. **Equilibrium Conditions**:

$$x_1^A + x_1^B = \omega_1^A + \omega_1^B$$
$$\frac{m^A}{p_1}\frac{\alpha}{\alpha+\beta} + \frac{m^B}{p_1}\frac{\gamma}{\gamma+\delta} = \omega_1^A + \omega_1^B$$

Replace m^A and m^B:

$$\frac{1}{p_1}\frac{\alpha}{\alpha+\beta}(p_1\omega_1^A + p_2\omega_2^A) + \frac{1}{p_1}\frac{\gamma}{\gamma+\delta}(p_1\omega_1^B + p_2\omega_2^B) = \omega_1^A + \omega_1^B$$

Considering $p_2 = 1$

$$\frac{1}{p_1}\frac{\alpha}{\alpha+\beta}(p_1\omega_1^A + \omega_2^A) + \frac{1}{p_1}\frac{\gamma}{\gamma+\delta}(p_1\omega_1^B + \omega_2^B) = \omega_1^A + \omega_1^B$$

We want to solve for p_1

$$p_1\omega_1^A\frac{\alpha}{\alpha+\beta} + \omega_2^A\frac{\alpha}{\alpha+\beta} + p_1\omega_1^B\frac{\gamma}{\gamma+\delta} + \omega_2^B\frac{\gamma}{\gamma+\delta} = (\omega_1^A + \omega_1^B)p_1$$
$$\omega_2^A\frac{\alpha}{\alpha+\beta} + \omega_2^B\frac{\gamma}{\gamma+\delta} = p_1(\omega_1^A + \omega_1^B - \omega_1^A\frac{\alpha}{\alpha+\beta} - \omega_1^B\frac{\gamma}{\gamma+\delta})$$
$$\omega_2^A\frac{\alpha}{\alpha+\beta} + \omega_2^B\frac{\gamma}{\gamma+\delta} = p_1\left(\omega_1^A\left(1-\frac{\alpha}{\alpha+\beta}\right) + \omega_1^B\left(1-\frac{\gamma}{\gamma+\delta}\right)\right)$$
$$\omega_2^A\frac{\alpha}{\alpha+\beta} + \omega_2^B\frac{\gamma}{\gamma+\delta} = p_1\left(\omega_1^A\frac{\beta}{\alpha+\beta} + \omega_1^B\frac{\delta}{\gamma+\delta}\right)$$
$$\frac{\omega_2^A\frac{\alpha}{\alpha+\beta} + \omega_2^B\frac{\gamma}{\gamma+\delta}}{\omega_1^A\frac{\beta}{\alpha+\beta} + \omega_1^B\frac{\delta}{\gamma+\delta}} = p_1^*$$

2. Given:

$$\alpha = \beta = \gamma = \delta = 1$$
$$\omega_1^A = 10$$
$$\omega_2^A = 15$$
$$\omega_1^B = 7$$
$$\omega_2^B = 20$$

We have: $\frac{p_1}{p_2} = \frac{35}{77}$ And:

$$m^A = P_1\omega_1^A + P_2\omega_2^A = 605$$
$$m^B = P_1\omega_1^A + P_2\omega_2^A = 585$$

Optimal demands:

$$x_1^A = \frac{1}{2} \cdot \frac{m^A}{p_1} = \frac{121}{14}$$
$$x_2^A = \frac{1}{2} \cdot \frac{m^A}{p_2} = \frac{605}{34}$$
$$x_1^B = \frac{1}{2} \cdot \frac{m^B}{p_1} = \frac{117}{14}$$
$$x_2^B = \frac{1}{2} \cdot \frac{m^B}{p_2} = \frac{585}{34}$$